PMM U.S.S.R.,Vol.47,No.3,pp.316-320,1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

UDC 531.38

ON THE STABILITY OF THE PERMANENT ROTATIONS OF AN ASYMMETRIC HEAVY RIGID BODY*

D.V. ANDREEV

The permanent rotations around the vertical of an asymmetric heavy rigid body with a fixed point are examined. The stability of the rotations are investigated on the basis of stability theorems for a Hamiltonian system in the nonresonance case /1,2/ and under third- and fourth-order resonances /3/. It is shown that in the nonresonance case the stability of all, except, perhaps, a finite number, permanent rotations is determined by the first approximation. Stability and instability conditions for resonance rotations are found. Stability of rotations, in the general case, was studied in /4 - 7/, in the case of rotations around the principal axes, in /8 - 10/, and in the case of rotations around axes lying in the principal inertia plane, in /11/.

1. We consider a heavy body with a fixed point O and with principal moments of inertia $A > B > C$ for this point. For simplicity we set the product of the body's weight by the distance from point O to the center of mass equal to unity. Let $Ox_1y_1z_1$ be a fixed coordinate system with the axis Oz_1 directed vertically upwards and let $Ox_0y_0z_0$ be a coordinate system connected with the body's principal axes of inertia for the point O If the body executes a permanent rotation around the vertical, then the relative position of these systems at the initial instant $t=0$ is specified by a table of direction cosines $\{n_{ij}\}$, where $n_{31}=\alpha$, $n_{32}=\beta$, $n_{33}=\gamma$ are constant through the whole rotation time. Together with system $Ox_0y_0z_0$ we introduce another system $Ox_1'y_1'z_1'$ rigidly attached to the body, where Oz_1' is the permanent axis in the body. At any instant the axes of this system are obtained from those of $Ox_1y_1z_1$ by successive turns through the angles: $\omega t+\psi$ around the z_1-axis, φ around the new position x_1'' of the x_1-axis, θ around the y_1'-axis. Under a permanent rotation of the body with angular velocity ω the axis Oz_1 coincides with the permanent axis Oz_1'. In this case the angles ψ, φ and θ equal zero and, consequently, are the Lagrange coordinates of the body in perturbed motion. We shall investigate the stability of the permanent axis, i.e., the stability with respect to the coordinates φ and θ; the coordinate $\omega t+\psi$ is cyclic.

We introduce the variables $u_1=\sin\varphi$, $u_2=\cos\varphi\sin\theta$, the dimensionless time $\Gamma=\omega t$ and the generalized momenta v_1, v_2. We construct the Hamiltonian H' of the variables u_1, u_2, v_1, v_2. The function H' has a stationary point $u_1=0$, $u_2=0$, $v_1=a_{23}$, $v_2=a_{13}$. Here and later $a_{ij}=An_{i1}n_{j1}+Bn_{i2}\cdot n_{j2}+Cn_{i3}n_{j3}$, $i,j=1,2,3$ are the components of the body's inertia tensor. We introduce the perturbations $u_3=v_1-a_{23}$, $u_4=v_2-a_{13}$ of the momenta and we consider the Hamiltonian $H=H'-H_0$, $H_0={}^3/_2a_{33}+\lambda$ of the canonic variables u_i, $i=1,\ldots,4$. We obtain the following expansion of H in a series in powers of u_i :

$$H=H_2+H_3+H_4+\ldots \tag{1.1}$$

$$H_2=\frac{1}{2}\sum_{i,j=1}^{4}r_{ij}u_iu_j,\quad H_3=\sum_{i+j+k+l=3}h_{ijkl}u_3^iu_4^ju_1^ku_2^l$$

$$H_4=\sum_{i+j+k+l=4}h_{ijkl}u_3^iu_4^ju_1^ku_2^l$$

and the following nonzero h_{ijkl} and r_{ij}:

$$r_{11}=a_{23}^2b_{33}+\lambda,\quad r_{12}=a_{23}(a_{33}b_{13}-a_{13}b_{33}),\ r_{22}=a_{13}(a_{33}b_{13}-a_{13}b_{33})+a_{33}(a_{33}b_{11}-a_{13}b_{13})+2a_{33}+\lambda,\quad r_{13}=-a_{23}b_{23}-1,\quad r_{14}=-a_{23}b_{13},\ r_{23}=a_{13}b_{23}-a_{33}b_{12},\quad r_{24}=1-a_{33}b_{11}+a_{13}b_{13},\ r_{33}=b_{22},\ r_{34}=b_{12},\ r_{44}=b_{11},\quad h_{2010}=h_{1101}=b_{23},$$

*Prikl.Matem.Mekhan.,Vol.47,No.3,pp.372-377,1983

$$h_{0201} = -h_{1110} = -b_{13},\quad h_{1002} = -h_{0111} = a_{13}b_{33} - a_{33}b_{13},$$
$$h_{1020} = a_{23}b_{33} + a_{33}b_{23},\ h_{1011} = -a_{23}b_{33},\ h_{0120} = a_{33}b_{13},\ h_{0030} = a_{23}(-a_{33}b_{33} + {}^1/_2),\ h_{0012} = {}^1/_2 a_{23},\ h_{0003} = -{}^1/_2 a_{13},\ h_{0021} = a_{33}h_{1002} - {}^1/_2 a_{33},\ h_{1120} = h_{1102} = -2b_{12},\ h_{0130} = h_{0112} = a_{23}b_{13},$$
$$h_{1021} = -a_{13}b_{23} - a_{33}b_{12},\ h_{1003} = a_{33}b_{12} - a_{13}b_{23},\ h_{2020} = b_{33} - b_{22},$$
$$h_{2002} = -b_{22},\quad h_{0220} = -b_{11},\quad h_{0202} = b_{33} - b_{11},\quad h_{1111} = -2b_{33},$$
$$h_{1030} = 1 + a_{23}b_{23} - 2a_{33}b_{13},\quad h_{1012} = 1 + a_{23}b_{23},\quad h_{0121} = -1 - a_{13}b_{13} - 2a_{33}b_{33},\ h_{0103} = a_{33}b_{11} - a_{13}b_{13} - 1,\ h_{0040} = a_{23}{}^2 b_{33} + {}^1/_4(\lambda - a_{33}),\ h_{0031} = 2a_{33}a_{23}b_{13},\ h_{0022} = 2a_{33}h_{0103} + {}^1/_2(\lambda - a_{33}),$$
$$h_{0004} = {}^1/_4(\lambda - a_{33})$$

Here

$$b_{ij} = (ABC)^{-2}(BCn_{i1}n_{j1}) + ACn_{i2}n_{j2} + ABn_{i3}n_{j3});\ i, j = 1, 2, 3$$

while the parameter λ is defined /5/ by the relations

$$\lambda = A - \alpha_0/\alpha = B - \beta_0/\beta = C - \gamma_0/\gamma \tag{1.2}$$

where $\alpha_0, \beta_0, \gamma_0$ are the direction cosines of the radius vector of the body's center of mass in the system $Ox_0y_0z_0$.

2. We consider the linearized system of equations of perturbed motion of the body, determined by Hamiltonian H_2. It has the operational matrix

$$\Delta(D) = \| r_{ij} \| + \mathbf{J}D,\quad \mathbf{J} = \begin{Vmatrix} 0 & \mathbf{E} \\ -\mathbf{E} & 0 \end{Vmatrix} \tag{2.1}$$

where $\mathbf{E}$ is the second-order unit matrix, and the characteristic equation

$$\sigma^4 + g_1\sigma^2 + g_2 = 0 \tag{2.2}$$
$$g_1 = r_{11}r_{33} - r_{13}{}^2 + r_{22}r_{44} - r_{24}{}^2 + 2(r_{12}r_{34} - r_{23}r_{14})$$
$$g_2 = \det \| r_{ij} \|$$

We shall reckon as fulfilled the necessary stability conditions /4/

$$g_1 > 0,\quad g_2 > 0,\quad g_3 = g_1{}^2 - 4g_2 > 0 \tag{2.3}$$

and consider the case when form H_2 is indefinite. In that case Hamiltonian H_2 is reduced to the normal form

$$H_2' = i/2\,(\omega_1 p_1 q_1 - \omega_2 p_2 q_2) \tag{2.4}$$

where $\omega_1 > \omega_2$ are the moduli of the roots of Eq.(2.2) and p_1, p_2, q_1, q_2 are the new canonic variables.

Let $F_{k1}(D)\ (k = 1, \ldots, 4)$ be the cofactor of the k-th algebraic element in the first row of matrix (2.1). We introduce the notation

$$f_{k1} = {}^1/_2 F_{k1}(i\omega_1)\,[\omega_1(\omega_2{}^2 - \omega_1{}^2)\,F_{11}(i\omega_1)]^{-1/2}$$
$$f_{k2} = {}^1/_2 F_{k1}(i\omega_2)\,[\omega_2(\omega_2{}^2 - \omega_1{}^2)\,F_{11}(i\omega_2)]^{-1/2}$$
$$(k = 1, \ldots, 4;\ i = \sqrt{-1})$$

The canonic transformation /12/

$$u_i = -f_{i1}p_1 + f_{i2}p_2 - \bar{f}_{i1}q_1 + \bar{f}_{i2}q_2\quad (i = 1, \ldots, 4)$$

normalizing H_2, where $\bar{f}_{kl}$ is the function complex-conjugate to f_{kl}, transforms H_3 and H_4 as well. In the new variables expansion (1.1) becomes

$$H = H_2' + \sum_{i+j+k+l=3} K_{ijkl}\,p_1{}^i p_2{}^j q_1{}^k q_2{}^l + \sum_{i+j+k+l=4} L_{ijkl}\,p_1{}^i p_2{}^j q_1{}^k q_2{}^l + \ldots \tag{2.5}$$

Here

$$K_{3000} = -\sum_{i+j+k+l=3} h_{ijkl}\,f_{11}{}^k f_{21}{}^l f_{31}{}^i f_{41}{}^j$$

while the expressions for the coefficients $K_{0300}, K_{0030}, K_{0003}$ are obtained from K_{3000} by the replacements of the sets

$$(f_{k1}, f_{k2}, \bar f_{k1}, \bar f_{k2}) \text{ by } (-f_{k2}, -f_{k1}, -\bar f_{k2}, -\bar f_{k1}) \tag{2.6}$$
$$(f_{k1}, f_{k2}, \bar f_{k1}, \bar f_{k2}) \text{ by } (\bar f_{k1}, \bar f_{k2}, f_{k1}, f_{k2})$$
$$(f_{k1}, f_{k2}, \bar f_{k1}, \bar f_{k2}) \text{ by } (-\bar f_{k2}, -\bar f_{k1}, -f_{k2}, -f_{k1})$$

Further

$$K_{2100} = \sum_{\nu_l=1,2,3,4} h_{k_1k_2k_3k_4}(f_{\nu_1 1} f_{\nu_2 1} f_{\nu_3 2} + f_{\nu_1 1} f_{\nu_2 2} f_{\nu_3 1} + f_{\nu_1 2} f_{\nu_2 1} f_{\nu_3 1})$$

(here and henceforth k_i is the number of indices ν_l taking value i). The expressions for the coefficients $K_{1200}, K_{0021}, K_{0012}$ are obtained from K_{2100} by the replacements (2.6); the expression for K_{2001} is obtained simply by replacing f_{k2} by $\bar f_{k2}$. The coefficients $K_{0210}, K_{0120}, K_{1002}$ are obtained by replacements (2.6) from the expression for K_{2001}. Having next replaced $\bar f_{k2}$ by $-\bar f_{k1}$, we obtain K_{2010} and K_{0120} from the coefficients K_{2001} and K_{1002}, respectively, while having replaces $\bar f_{k1}$ by $-\bar f_{k2}$ in the expressions for K_{0120} and K_{0210}, we obtain K_{0102} and K_{0201}. Finally,

$$K_{1110} = \sum_{\nu_l=1,2,3,4} h_{k_1k_2k_3k_4} \sum f_{\nu_1 1} f_{\nu_2 1} f_{\nu_3 1}$$

where the inner summation is taken over all permutations of indices ν_1, ν_2, ν_3. The coefficients $K_{1101}, K_{1011}, K_{0111}$ are obtained by replacements (2.6) from K_{1110}.

For the normalization of Hamiltonian (2.5) in the nonresonance case we need as well $L_{2020}, L_{0202}, L_{1111}$. We have

$$L_{2020} = \frac{1}{2} \sum_{\nu_l=1,2,3,4} h_{k_1k_2k_3k_4} \sum \bar f_{\nu_1 1} \bar f_{\nu_2 1} f_{\nu_3 1} f_{\nu_4 1}$$

where the inner summation is taken over all distinct permutations of the conjugacy symbols over the functions f_{kl}. The coefficient L_{0202} is obtained from L_{2020} by replacing f_{k1} by f_{k2} and $\bar f_{k1}$ by $\bar f_{k2}$. Finally

$$L_{1111} = \sum_{\nu_l=1,2,3,4} h_{k_1k_2k_3k_4} \sum f_{\nu_1 1} \bar f_{\nu_2 1} \bar f_{\nu_3 2} f_{\nu_4 2}$$

where the inner summation is taken over all possible permuations of the indices $\nu_1, \nu_2, \nu_3, \nu_4$.

3. If neither one of the conditions

$$\omega_1 = 2\omega_2, \quad \omega_1 = 3\omega_2 \tag{3.1}$$

is fulfilled, Hamiltonian (2.5) can be reduced to the form

$$H = H_2' + G_{11} p_1^2 q_1^2 + 2G_{12} p_1 p_2 q_1 q_2 + G_{22} p_2^2 q_2^2 + \ldots$$

According to /13/

$$\begin{aligned} G_{11} &= L_{2020} + \omega_1^{-1}(K_{3000}K_{0300} + 3K_{2010}K_{1020}) + 3\omega_2^{-1} K_{1110}K_{1011} + \\ &(4\omega_1 + 3\omega_2)(2\omega_1 + \omega_2)^{-2} K_{2100}K_{0012} + \\ &(3\omega_1 - 4\omega_2)(2\omega_1 - \omega_2)^{-2} K_{0120}K_{2010} \\ G_{12} &= L_{1111} + 2\omega_1 (2\omega_1 + \omega_2)^{-2} K_{2100}K_{0021} + 2\omega_2 (2\omega_2 + \omega_1)^{-2} \times \\ &K_{1200}K_{0012} + 2\omega_1 (2\omega_1 - \omega_2)^{-2} K_{2001}K_{0120} + 2\omega_2 (2\omega_2 - \\ &\omega_1)^{-2} K_{0210}K_{1002} + 2\omega_1^{-1}(K_{2010}K_{0111} + K_{1101}K_{1020}) + \\ &2\omega_2^{-1}(K_{1110}K_{0102} + K_{0201}K_{1011}) \end{aligned}$$

G_{22} is obtained from G_{11} as follows: ω_1 changes places with ω_2, in the multi-indices the first indices permute with the second and the third with the fourth. On the Stäude cone /14/ and on the circle of centers of gravity /4/ we try to pick out axes for which

$$W = G_{11}\omega_1^2 + 2G_{12}\omega_1\omega_2 + G_{22}\omega_2^2 = 0 \tag{3.2}$$

According to /1– 3/, rotations around axes lying in the gyroscopic stability domains /4,6,7/ will be stable if neither one of the conditions (3.1) and (3.2) is fulfilled for them. It is well known /5– 7/ that with each "allowable" axis on the Stäude cone we can associate one-to-one a value of the real parameter λ defined by relations (1.2). It can be shown that the circle of centers of gravity, as also the Stäude cone, can be specified by parameter λ if we reckon constant not $\alpha_0, \beta_0, \gamma_0$, but α, β, γ.

We return to condition (3.2). This is an irrational equation in λ. By getting rid of the radicals it reduces to an algebraic one. When the α, β, γ are fixed, its degree equals 176, while for constants $\alpha_0, \beta_0, \gamma_0$, it does not exceed 28,512. Therefore, the number of axes for which the Arnol'd-Moser determinant $W(\lambda)$ vanishes on the whole Stäude cone or on the circle of centers of gravity, and, hence, in the gyroscopic stability domains, is a finite number.

4. We pass on to the resonance case and we consider rotations for which one of conditions (3.1) is fulfilled. We express the squares ω_1^2, ω_2^2 of the frequencies in terms of the coefficients of Eq.(2.2) and we substitute these expressions into (3.1). Equations (3.1) take the form

$$g_3 + g_1\sqrt{g_3} = 6g_2, \quad g_3 + g_1\sqrt{g_3} = 16g_2 \tag{4.1}$$

It can be shown that each of the Eqs.(4.1) has exactly one root in the gyroscopic stability domains for fixed α, β, γ, while for constants $\alpha_0, \beta_0, \gamma_0$, no more than 20 roots. Consequently, to each of the third- and fourth-order resonance relations there corresponds one point each on the circle of centers of gravity and no more than 20 axes on the Stäude cone.

In the case of third-order resonance the Hamiltonian (2.5) is reduced to the form /13/

$$\begin{aligned} &H = H_2' + K_{1002}p_1q_2^2 + \bar{K}_{1002}p_2^2q_1 + \\ &G_{11}p_1^2q_1^2 + 2G_{12}'p_1p_2q_1q_2 + G_{22}'p_2^2q_2^2 + \ldots \\ &K_{1002} = -\sum_{\nu_j=1,2,3,4} h_{k_3k_4k_1k_2}(f_{\nu_1 1}\bar{f}_{\nu_2 2}\bar{f}_{\nu_3 2} + \\ &\quad \bar{f}_{\nu_1 2}f_{\nu_2 1}\bar{f}_{\nu_3 2} + \bar{f}_{\nu_1 2}\bar{f}_{\nu_2 2}f_{\nu_3 1}) \\ &G_{12}' = G_{12} - 2\omega_2(\omega_1 - 2\omega_2)^{-2}K_{1002}\bar{K}_{1002} \\ &G_{22}' = G_{22} - (3\omega_2 - 4\omega_1)(\omega_1 - 2\omega_2)^{-2}K_{1002}\bar{K}_{1002} \end{aligned} \tag{4.2}$$

The relations /3/

$$K_{1002} = 0, \quad G_{11} + 4G_{12}' + 4G_{22}' \neq 0 \tag{4.3}$$

are the stability conditions for the corresponding rotations, and, moreover, the first one of them is necessary. To obtain the stability conditions for a concrete resonance rotation the coefficients must be computed for values of the parameter λ which are corresponding roots of the first of Eqs.(4.1).

In the case of fourth-order resonance the Hamiltonian (2.5) is transformed to

$$\begin{aligned} H &= H_2' + G_{0310}q_1p_2^3 + \bar{G}_{0310}p_1q_2^3 + G_{11}p_1^2q_1^2 + 2G_{12}p_1p_2q_1q_2 + \\ &G_{22}p_2^2q_2^2 + \ldots, \; G_{0310} = L_{0310} + 2(2\omega_1 - \omega_2)^{-1}K_{1200}K_{0120} + \\ &(\omega_1 - 2\omega_2)^{-1}K_{0210}(K_{1110} - 2K_{0201}) + 2\omega_1(\omega_1 + 2\omega_2)^{-1} \times \\ &(2\omega_1 - \omega_2)^{-1}K_{0120}K_{1200} + \omega_1\omega_2^{-1}(\omega_1 - 2\omega_2)^{-1}K_{0210}K_{1110} + \\ &\omega_2^{-1}(K_{0111}K_{0300} + K_{0210}K_{1110}) \end{aligned} \tag{4.4}$$

When the two inequalities /3/

$$G_{0310} \neq 0, \quad \omega_1 |G_{0310}| \geqslant |G_{11} + 6G_{12} + 9G_{22}| \tag{4.5}$$

are fulfilled simultaneously the permanent rotations corresponding to fourth-order resonance will be unstable; in the case of opposite sign in the second inequality, they are stable. Stability is preserved if the right-hand side of the second of inequalities (4.5) is nonzero when $G_{0310} = 0$.

The coefficients occurring in formulas (4.3) and (4.5) as functions of the rigid body's parameters, for each fixed value of λ, are not identically zero. Consequently, conditions (4.3), (4.5) impose constraints only of the rigid body's mass distribution. Thus, the stability of all (except, possibly, a finite number) nonresonance rotations is determined by the first approximation. The stability of instability of the resonance rotations is determined by coefficients (4.2) and (4.4) from conditions (4.3) and (4.5) and depends only on the rigid body's mass distribution.

The author thanks V.V. Rumiantsev for attention to the work.

REFERENCES

1. ARNOL'D V.I., Small denominators and stability problems in classical and celestial mechanics. Uspekhi Matem. Nauk, Vol.18, No.6, 1963.

2. MOSER J.K., Lectures on Hamiltonian Systems. Providence, RI. American Mathematical Society, 1968.
3. MARKEEV A.P., On stability of a canonical system with two degrees of freedom in the presence of resonance. PMM Vol.32, No.4, 1968.
4. GRAMMEL'R., Der Kreisel, seine Theorie und seine Anwendungen. Zweite, neubearbeitete Auflage. Erster Band: Die Theorie des Kreisels. Berlin-Göttingen-Heidelberg, Springer-Verlag, 1950. (The Gyroscope, Its Theory and Application, Vol.1).
5. RUMIANTSEV V.V., Stability of permanent rotations of a heavy rigid body. PMM Vol.20, No.1, 1956.
6. RUBANOVSKII V.N., On bifurcations and stability of steady-state motions of systems with known first integrals. In: Problems in the Investigation of Stability and Motion Stabilization. Moscow, Vychisl. Tsentr Akad. Nauk SSSR, No.1, 1975.
7. RUBANOVSKII V.N., On bifurcations and stability of permanent rotations of a rigid body in the case when its center of mass lies close to the principle inertia plane. In: Problems in the Investigation of Stability and Motion Stabilization. Moscow, Vychisl. Tsentr Akad. Nauk SSSR, 1982.
8. KOVALEV A.M. and SAVCHENKO A.Ia., Stability of uniform rotations of a rigid body about a principal axis. PMM Vol.39, No.4, 1975.
9. SERGEEV V.S., On the stability of the permanent rotations of a heavy solid body around a fixed point. PMM Vol.40, No.3, 1976.
10. CHUDNENKO A.N., On stability of the uniform rotations of a rigid body around the principal axis. PMM Vol.44, No.2, 1980.
11. BELIKOV S.A., On the stability of permanent rotations of a rigid body around nonprincipal axis in a Newtonian force field. Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, No.1, 1982.
12. BULGAKOV B.V., On normal coordinates. PMM Vol.10, No.2, 1946.
13. BIRKHOFF G.D., Dynamical Systems. Providence, RI, American Mathematical Society, 1966.
14. STÄUDE O., Über permanente Rotationsachen bei der Bewegung eines schweren Körpers um einen festen. Punkt. J. reine und angew. Mathe., B.113, 1894.

Translated by N.H.C.